On robust pricing–hedging duality in continuous time

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based on joint work with

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Outline

B & I: Robust framework

INPUTS:

- Beliefs and Information:
  Prices of risky assets (underlying and some options) $(S_t^i)_{t \le T}$,
  $i = 0, 1, \ldots, |\mathbb{K}|$ *belong to some path space* $\mathfrak{P} \subseteq \mathcal{I}$. This encodes
  - Information ($\mathcal{I}$): e.g. today's prices, future payoff restrictions.
  - Beliefs ($\mathfrak{P}$): about feasible future prices

  Options $X \in \mathcal{X}$, *with known prices* $\mathcal{P}(X)$
- Rules: *no frictions,*
  dynamic trading in underlying plus selected options
  trading restrictions on $\mathcal{X}$, *e.g. buy-and-hold only*

REASONING PRINCIPLES:

- *no-arbitrage* $\iff$ *exists a* $\mathfrak{P}$*–market model* $\iff$ *efficient beliefs*

OUTPUTS:

- no arbitrage prices of G $\quad\Leftrightarrow\quad$ $LB \le \mathrm{Price}(G) \le UB$
- P-H duality: $\sup_{\mathfrak{P}\text{–market models}} \mathbb{E}[G] = \inf\{\text{superhedging cost}\}$

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An active field of research...

Explicit bounds $LB \leq \mathcal{P}O_T \leq UB$ and robust super-/sub- hedges in:

Hobson (98); Brown, Hobson and Rogers (01), Dupire (05), Lee (07), Cox, Hobson and O. (08), Cox and O. (11,11), Cox and Wang (12), Hobson and Klimmek (12,13), Galichon et al. (14), O. and Spoida (14),...

Arbitrage considerations and robust FTAP in:

Davis and Hobson (07), Cox and O. (11,11) and Davis, O. and Raval (13), Acciaio et al. (13), Bouchard and Nutz (14), Burzoni, Frittelli and Maggis (14), and ongoing ...

Pricing-hedging duality in:

Davis, O. and Raval (13), Beiglböck, Henry-Labordère and Penkner (13), Neufeld and Nutz (13), Dolinsky and Soner (13), Tan and Touzi (13), Galichon et al. (14), Bouchard and Nutz (14), Possamaï et al. (14), Fahim and Huang (14), Cox, Hou and O. (14), and ongoing...

Pathspace restrictions $\mathfrak{P} \subsetneq \Omega$:

Mykland (01,05), O. and Spoida (14), Hou and O. (14), Nadtochiy and O. (14), ...

The proposed framework

"Universally acceptable" starting point:

- no frictions
- risky assets: d stocks & N_c European options with information
  $\mathcal{I} := \{\omega \in C([0,T],\mathbb{R}_+) : \omega_0 = 1, \omega_T^{(i)} = h_i(\omega_T^{(1)},\dots,\omega_T^{(d)})\ \forall i \leq d\}$
- traded continuously using

$$\mathcal{A} = \Big\{\Delta : \mathcal{I}\times[0,T] \to \mathbb{R}^{d+N} \text{ simple (or of f.v.)}, \int_0^{\cdot} \Delta_u(\omega)\mathrm{d}\omega_u \geq -M \\ \text{for some } M \text{ and } \Delta_t(\omega) = \Delta_t(\omega') \text{ for } \omega_{|[0,t]} = \omega'_{|[0,t]}\Big\}$$

- options $X \in \mathcal{X}$ available for static trading, with prices $\mathcal{P}(X)$.

Beliefs: The pathspace (or prediction set) $\mathfrak{P} \subset \mathcal{I}$.

Superhedging price: $V_{\mathcal{I},\mathcal{X},\mathcal{P}}(G) :=$

$$\inf\left\{\sum_{i=0}^{K} a_i\mathcal{P}(X_i) : \exists \Delta \in \mathcal{A}, X_i \in \mathcal{X} \text{ s.t. } \sum_{i=0}^{K} a_i X_i(\omega) + \int_0^T \Delta_t \mathrm{d}\omega_t \geq G(\omega)\ \forall \omega \in \mathcal{I}\right\}$$

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Probabilistic objects:

- a $(\mathcal{X}, \mathcal{P}, \mathfrak{P})$-model (or market model) is a martingale measure $\mathbb{P}$ on $\mathcal{I}$ such that $\mathbb{P}(\mathfrak{P}) = 1$ and $\mathbb{E}^{\mathbb{P}}[X] = \mathcal{P}(X)\ \forall X \in \mathcal{X}$.
- Relaxation: a $(\mathcal{X}, \mathcal{P}, \mathfrak{P}, \eta)$-model (or η-market model) is a martingale measure $\mathbb{P}$ on $\mathcal{I}$ such that $\mathbb{P}(\mathfrak{P}^\eta) \geq 1 - \eta$ and $|\mathbb{E}^{\mathbb{P}}[X] - \mathcal{P}(X)| \leq \eta\ \forall X \in \mathcal{X}$, where $\mathfrak{P}^\eta := \{\omega \in \mathcal{I} : \inf_{v \in \mathfrak{P}} \|\omega - v\| \leq \eta\}$.

Pricing–Hedging relation: for $G : \mathcal{I} \to \mathbb{R}$ and a super-replicating strategy

$$\mathbb{E}^{\mathbb{P}}[G(\omega)] \leq \mathbb{E}^{\mathbb{P}}\left[\sum_{i=0}^{K} a_i X_i(\omega) + \int_0^T \Delta_t \mathrm{d}\omega_t\right] = \sum_{i=0}^{K} a_i \mathcal{P}(X_i)$$

and hence

$$P_{\mathcal{X},\mathcal{P},\mathfrak{P}}(G) := \sup_{\mathbb{P}:(\mathcal{X},\mathcal{P},\mathfrak{P})\text{-model}} \mathbb{E}^{\mathbb{P}}[G] \leq V_{\mathcal{X},\mathcal{P},\mathfrak{P}}(G).$$

Similarly

$$\widetilde{P}_{\mathcal{X},\mathcal{P},\mathfrak{P}}(G) := \lim_{\eta \searrow 0} \sup_{\mathbb{P}:(\mathfrak{P},\mathcal{X},\eta)\text{-model}} \mathbb{E}^{\mathbb{P}}[G] \leq \lim_{\eta \searrow 0} V_{\mathcal{X},\mathcal{P},\mathfrak{P}^\eta}(G) =: \widetilde{V}_{\mathcal{X},\mathcal{P},\mathfrak{P}}(G).$$

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and hence (**Full Duality:**)

$$P_{\mathcal{X},\mathcal{P},\mathfrak{P}}(G) := \sup_{\mathbb{P}:(\mathcal{X},\mathcal{P},\mathfrak{P})\text{-model}} \mathbb{E}^{\mathbb{P}}[G] \stackrel{?}{=} V_{\mathcal{X},\mathcal{P},\mathfrak{P}}(G).$$

Similarly (**Approximate Duality:**)

$$\widetilde{P}_{\mathcal{X},\mathcal{P},\mathfrak{P}}(G) := \lim_{\eta \searrow 0} \sup_{\mathbb{P}:(\mathfrak{P},\mathcal{X},\eta)\text{-model}} \mathbb{E}^{\mathbb{P}}[G] \stackrel{?}{=} \lim_{\eta \searrow 0} V_{\mathcal{X},\mathcal{P},\mathfrak{P}^\eta}(G) =: \widetilde{V}_{\mathcal{X},\mathcal{P},\mathfrak{P}}(G).$$

Examples

Dolinsky and Soner (13)/Martingale optimal transport:
Take $d = 1$, $N = 0$ so that $\mathcal{I} = \{\omega \in C([0,T], \mathbb{R}_+) : \omega_0 = 1\}$ and $\mathcal{X} = \{(\omega_T - K)^+/C(K) : K \geq 0\}$. Then

$$\sup_{\mathbb{P}:(\mathcal{X},\mathcal{P},\mathcal{I})\text{-model}} \mathbb{E}^{\mathbb{P}}[G] = V_{\mathcal{I}}(G) \quad \forall \text{ bd and unif. cont.} G.$$

$(\mathcal{X}, \mathcal{P}, \mathcal{I})$-models are martingale measures with $\omega_T \sim \mu$, as

$$(\mathcal{X}, \mathcal{P}, \mathcal{I})\text{-model} \equiv \mathbb{E}_{\mathbb{Q}}[(\mathbb{S}_T - K)^+] = C(K), \quad \forall K$$
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Pricing-hedging duality (without beliefs)

Setup:

- risky assets: d stocks and N_c options traded continuously
- $N_s + N_c$ options in $\mathcal{X}$ for static trading

Assumptions:

- all payoffs are bounded and uniformly continuous
- there exists an $(\mathcal{X}, \tilde{\mathcal{P}}, \mathcal{I})$-model for every $\tilde{\mathcal{P}}$ such that $|\tilde{\mathcal{P}}(X) - \mathcal{P}(X)| \le \eta \, \forall X \in \mathcal{X}$ when η is small

Theorem

For any bounded uniformly continuous $G : \mathcal{I} \to \mathbb{R}$

$$V_{\mathcal{X},\mathcal{P},\mathcal{I}}(G) = P_{\mathcal{X},\mathcal{P},\mathcal{I}}(G) := \sup_{\mathbb{P}:(\mathcal{X},\mathcal{P},\mathcal{I})\text{-model}} \mathbb{E}^{\mathbb{P}}[G]$$

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Pricing-hedging duality (with beliefs)

Setup:

- risky assets as before
- beliefs given via $\mathfrak{P} \subset \mathcal{I}$

Assumptions:

- all payoffs are bounded and uniformly continuous
- $\mathrm{Lin}_1(\mathcal{X}) := \left\{ a_0 + \sum_{i=1}^m a_i X_i \;:\; m \in \mathbb{N},\, X_i \in \mathcal{X},\, \sum_{i=0}^m |a_i| \le 1 \right\}$ is a compact subset of $\mathcal{C}(\mathcal{I}, \mathbb{R})$
- for all $\eta > 0$ there exists a $(\mathcal{X}, \mathcal{P}, \mathfrak{P}, \eta)$-model

Theorem

For any bounded uniformly continuous $G : \mathcal{I} \to \mathbb{R}$

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where $\tilde{V}_{\mathcal{X},\mathcal{P},\mathfrak{P}}(G) = \inf\{$*superhedging cost of G on $\mathfrak{P}^\eta$ for some η*$\}$.

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(MOT) Pricing-hedging duality (with beliefs)

Setup:

- d stocks trade continuously ($N_c = 0$)
- $\mathcal{X}$ = call options with n maturities and all strike trade statically
- beliefs given via $\mathfrak{P} \subset \mathcal{I}$

Assumptions:

- for all η there exists a martingale measure $\mathbb{P} \in \mathcal{M}_{\vec{\mu},\mathfrak{P},\eta}$ s.t. $\mathbb{P}$
  - reprices calls with maturity T_n
  - η-reprices calls with maturity T_i, $i < n$, and
  - $\mathbb{P}(\mathfrak{P}^\eta) > 1 - \eta$

Theorem

For any bounded uniformly continuous $G : \mathcal{I} \to \mathbb{R}$

$$\tilde{V}_{\mathcal{X},\mathcal{P},\mathfrak{P}}(G) = \lim_{\eta \searrow 0} \sup_{\mathbb{P} \in \mathcal{M}_{\vec{\mu},\mathfrak{P},\eta}} \mathbb{E}^{\mathbb{P}}[G].$$

(MOT) Pricing-hedging duality (with beliefs)

Setup:

- d stocks trade continuously ($N_c = 0$)
- $\mathcal{X}$ = call options with n maturities and all strike trade statically
- beliefs given via $\mathfrak{P} \subset \mathcal{I}$

Assumptions:

- for all η there exists a martingale measure $\mathbb{P} \in \mathcal{M}_{\vec{\mu},\mathfrak{P},\eta}$ s.t. $\mathbb{P}$
  - reprices calls with maturity T_n
  - η-reprices calls with maturity T_i, $i < n$, and
  - $\mathbb{P}(\mathfrak{P}^\eta) > 1 - \eta$

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Thank You!